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AN AXI-SYMMETRIC INFINITE ELEMENT FOR TRANSIENT RADIAL FLOW PROBLEMS

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SUMMARY

The formulation of an axi-symmetric infinite element for transient analysis of flow problems in unbounded domain is presented. The theoretical basis as well as the implementation of the element is discussed, and the element decay function is derived using the analytical solution of a one-dimensional axially symmetric configuration. The form of decay within the element is described as a function of both time and space, and thus the hydraulic head distribution in the far field is simulated rigorously. The accuracy and the efficiency of the proposed element are demonstrated through several numerical examples in infinite media. In general, it is shown that using the present infinite element transient flow problems in unbounded domains can be simulated effectively. Copyright © 1999 John Wiley & Sons, Ltd.

Key words: infinite element; groundwater; transient; unbounded domain

1. INTRODUCTION

Groundwater engineering problems often involve flow domains, which have a finite thickness but are laterally extensive. The common engineering approach in numerical solution of these problems is to truncate the remote boundary at a large distance from the zone of interest and then impose free or fixed boundary conditions. The imposition of such boundaries, however, can lead to spurious solutions particularly if the truncation occurs too close to the zone of interest. On the other hand, a large number of finite elements may be used to model the remote domain, which can lead to high computational costs, storage and time penalties.

In the past, a number of approaches have been proposed to deal with unbounded domains. Typical examples are the boundary integral method[1–3], energy absorbing boundaries,[4] techniques based on the eigenvalue solution of the problem,[5,6] and the cloning technique.[7–9] Also utilized extensively has been the concept of finite inelements.[10–12] These elements, which extend to infinity in one or more directions, are used in conjunction with the conventional finite elements such that the near field is represented by the finite elements and the far field by the infinite

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elements. The formulation of infinite elements is essentially the same as that for the finite elements except that their shape function (also referred to as the 'decay function') is selected on the basis of the far field behaviour of the analytical solution to the problem of interest. Infinite elements are easily coupled with finite elements and possess all inherent advantages of finite elements such as the banded and symmetric nature of the stiffness matrix and the versatility in simulating complex geometry.

To date, the concept of infinite elements has been applied to many engineering problems such as, thermal and groundwater flow,[13–17] ground freezing,[18] consolidation,[19,20] elasto-statics,[21] underground openings,[22,23] electromagnetics[24] and dynamic problems in both solid[25–28] and saturated poro-elastic media.[29] Zienkiewicz and Bettess[16] applied infinite elements to ocean wave propagation in unbounded domains. Wood[17] applied infinite elements to steady-state groundwater problems and Bettess[11] used infinite elements to solve viscous fluid flow. Askar and Lynn[18] and later Damjanic and Owen[14] applied steady-state infinite elements to transient thermal flow problems, which in fact may only be applicable to obtain the large time, steady-state response. The use of steady-state infinite elements in transient flow problems was also investigated by Honjo and Pokharel.[12] They analysed a number of flow problems in one, two and three dimensions and showed that the use of steady-state infinite elements in transient flow problems would only improve the results marginally (i.e. by one log-cycle in the time domain), and that the effect of truncated boundaries cannot be avoided after a long period of time (see Figure 1).

It should be noted that in transient problems, for each increase in time, an additional part of the unbounded domain is incorporated into the domain of influence, requiring the property matrices of the infinite element to be a function of time. Such time dependency of the property matrices

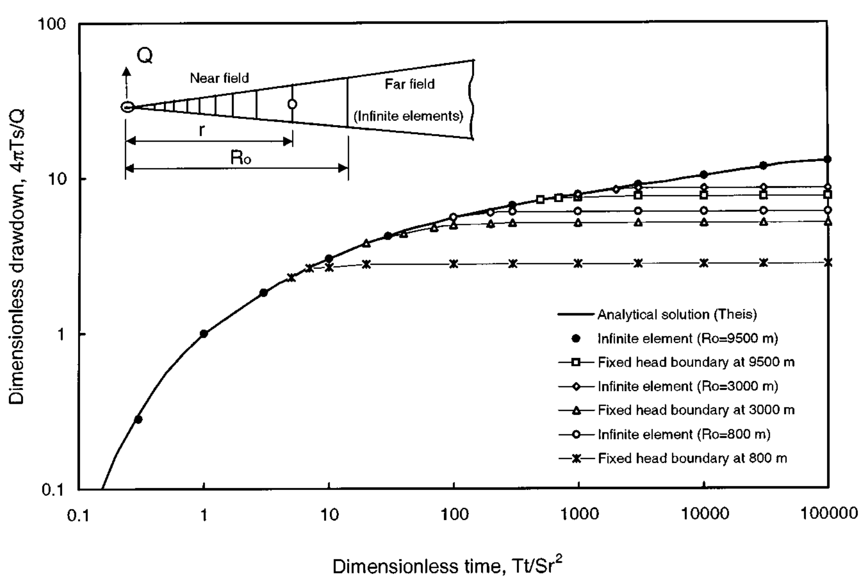

Figure 1. Comparison of results obtained using steady-state infinite elements and analytical solution (Theis) for axi-symmetric radial flow at $r = 200$ (after Honjo and Pokharel)

cannot be taken into account using steady-state infinite elements. To overcome this problem, Zhao and Valliappan[15] for the first time introduced the concept of transient infinite elements. In these elements, the form of decay is defined as a function of both time and space, and thus the hydraulic head distribution within the element is simulated rigorously. They applied the concept to a number of flow problems in the infinite plane and reported excellent results in all cases.

In this paper, the concept of transient infinite elements is extended to axi-symmetric radial flow problems in unbounded domains. A detailed description of the element along with the derivation of the element decay function as well as the property matrices of the element is presented, based on the analytical solution of a one-dimensional axially symmetric configuration. The application and the effectiveness of the proposed infinite element are demonstrated through several radial flow problems in infinite domain.

2. FINITE ELEMENT FORMULATION

For the case of axially symmetric two dimensional radial flow, the governing equations in cylindrical co-ordinates for a general anisotropic case is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(T_{rr}r\frac{\partial h}{\partial r}\right)+T_{zz}\frac{\partial^2 h}{\partial z^2}=S\frac{\partial h}{\partial t} \tag{1}$$

in which, h is the total head; T_{rr} and T_{zz} are the principal transmissivities in the radial and vertical directions; S is the storativity; r and z are the radial and vertical coordinate directions, respectively; and t is the time.

Using the Galerkin weighted residual approach, the discretized form of (1) can be written as

$$[K]\{h\}+[M]\left\{\frac{\partial h}{\partial t}\right\}=\{f\} \tag{2}$$

where $\{h\}$ is the total head vector; $\{f\}$ is the flux vector; and $[K]$ and $[M]$ are the global matrices of the system. The element matrices can be expressed as

$$\begin{aligned}
[K]^e &= \int_{\Omega^e}\left(T_{rr}\frac{\partial[N]^T}{\partial r}\frac{\partial[N]}{\partial r}+T_{zz}\frac{\partial[N]^T}{\partial z}\frac{\partial[N]}{\partial z}\right)d\Omega^e\\
[M]^e &= \int_{\Omega^e} S[N]^T[N]\,d\Omega^e\\
\{f\}^e &= \int_{\Gamma^e}[N]^T q\,d\Gamma^e
\end{aligned} \tag{3}$$

where Ω^e is the element domain, Γ^e is the traction boundary of the element, q is the flux normal to the element boundary, and $[N]$ is the shape function matrix of the element. It is worth noting that equation (3) is applicable to both finite elements and infinite elements except for different shape function matrices.

3. INFINITE ELEMENT FORMULATION

The formulation of an infinite element generally consists of two main steps: (1) identification of the fundamental solution to the problem of interest, and (2) derivation of the shape function from the fundamental solution.[29]

For axially symmetric flow, the analytical solution for the problem of a pumping well fully intesecting a laterally extensive confined aquifer is available in the form[13]

$$s(u) = \frac{Q}{4\pi T}\int_u^{\infty} \frac{e^{-u}}{u}\,du \tag{4}$$

with

$$u = r^2 S/4Tt \tag{5}$$

where Q represents the pumping rate and s is the drawdown. This solution, which is known as the Theis solution in groundwater engineering, can also be written as a convergent series:

$$s(u) = \frac{Q}{4\pi T}\left[-0{\cdot}5772 - \ln u + u - \frac{u^2}{2\times 2!} + \frac{u^3}{3\times 3!} - \frac{u^4}{4\times 4!} + \cdots\right] \tag{6}$$

For small values of u, or for the large time response, solution (6) may be truncated after the first two terms. The large time response of the solution will be of particular interest for the purpose of this investigation, to obtain the far field behaviour and thus to derive the form of decay within the proposed infinite element. It is to be noted that, in transient problems, the drawdown will take some time before reaching the far field, and thus only the large time response of the solution may need to be considered to derive the form of decay within the infinite element.

At large time, the drawdown may be expressed by the asymptote

$$s(r, t) = \frac{Q}{4\pi T}\ln\frac{2{\cdot}25Tt}{r^2 S} \tag{7}$$

Now, to obtain the form of decay within the infinite element, consider the axially symmetric infinite element shown in Figure 2. The element is subjected to a pumping flux of $Q = 1$ at the origin of the global co-ordinate system; the global co-ordinate of node 1 (in the r direction) is r_0 and the local co-ordinate of the node is zero. Using the solution given in (7), the drawdown for a given time, t, at node 1 can be expressed as

$$s(r_0, t) = \frac{1}{4\pi T}\ln\frac{2{\cdot}25Tt}{r_0^2 S} \tag{8}$$

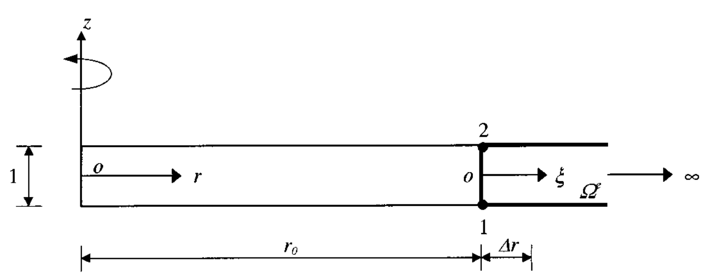

Figure 2. Global and local system of co-ordinates for the infinite element

Similarly, for any point within the infinite element, taking $r = r_0 + \Delta r$ as an example, the drawdown can be written as

$$s(r_0 + \Delta r, t) = \frac{1}{4\pi T} \ln \frac{2{\cdot}25Tt}{(r_0 + \Delta r)^2 S} \tag{9}$$

which may be rearranged to

$$s(r_0 + \Delta r, t) = s(r_0, t) \frac{\ln \dfrac{2{\cdot}25Tt}{(r_0 + \Delta r)^2 S}}{\ln \dfrac{2{\cdot}25Tt}{r_0^2 S}} \tag{10}$$

Noting that $\xi = \Delta r$, the decay function for the element can be expressed as

$$F(\xi, t) = \frac{\ln \dfrac{2{\cdot}25Tt}{(r_0 + \xi)^2 S}}{\ln \dfrac{2{\cdot}25Tt}{r_0^2 S}} \tag{11}$$

The total head within the infinite element may then be described as

$$h(\xi, t) = h_1 F(\xi, t) \tag{12}$$

where h_1 is the total head at Node 1. However, it is to be noted that function (11) approaches ∞ as $\xi \to \infty$. Therefore, in its current form, it cannot be used as a shape function of the proposed infinite element. To satisfy the boundary conditions at infinity, the shape function must decay with distance and approach zero as the distance approaches infinity. Furthermore, the integrations of the function over the element must lead to finite solutions. To satisfy these requirements, the decay function for the infinite element is expressed as

$$\hat{F}(\xi, t) = \frac{\ln (1 + 2{\cdot}25Tt/(r_0 + \xi)^2 S)}{\ln (1 + 2{\cdot}25Tt/r_0^2 S)} \tag{13}$$

which has the same logarithmic decay as function (11), except that it has a value of 1 at $\xi = 0$ and a value of zero as $\xi \to \infty$. Furthermore, it leads to integrations over the element, which are finite. It may also be noted that for the large time response, i.e. $Tt/r^2 S \geqslant 10$, the function $\ln(1 + 2{\cdot}25Tt/r^2 S)$ will have essentially the same numerical value as function $\ln(2{\cdot}25Tt/r^2 S)$. As can be observed, decay function (13) depends not only on the transmisivity and storativity of the medium, but also on the time and the space variables in the analysis. In fact, it is this consideration of the effect of the time variable which signifies the importance of the proposed infinite element for axially symmetric flow problems.

Using (13), the shape function for a laterally extensive two-dimensional axi-symmetric infinite element may be written as

$$[N] = [N_1, N_2] \tag{14}$$

where

$$N_1 = \hat{F}(\xi, t)\frac{1-\eta}{2}, \quad N_2 = \hat{F}(\xi, t)\frac{1+\eta}{2} \tag{15}$$

where η being the local co-ordinate axis in the vertical direction, having a value of -1 at node 1 and $+1$ at node 2.

Now, taking $\mathrm{d}r = \mathrm{d}\xi$ and $\mathrm{d}z = \frac{1}{2}(z_2 - z_1)\,\mathrm{d}\eta$ and using the shape function given in (14) the property matrices of the infinite element can be expressed as

$$[K]^{\mathrm{e}} = \int_0^\infty \int_{-1}^{1} \left(T_{rr} \frac{\partial [N]^{\mathrm{T}}}{\partial \xi} \frac{\partial [N]}{\partial \xi} + \frac{4T_{zz}}{(z_2 - z_1)^2} \frac{\partial [N]^{\mathrm{T}}}{\partial \eta} \frac{\partial [N]}{\partial \eta} \right) \frac{(z_2 - z_1)}{2}\, 2\pi (r_0 + \xi)\,\mathrm{d}\xi\,\mathrm{d}\eta$$

$$[M]^{\mathrm{e}} = \int_0^\infty \int_{-1}^{1} S\,[N]^{\mathrm{T}}[N] \frac{(z_2 - z_1)}{2}\, 2\pi (r_0 + \xi)\,\mathrm{d}\xi\,\mathrm{d}\eta \tag{16}$$

Equation (16) can be further reduced to

$$[K]^{\mathrm{e}} = \frac{T_{rr}(z_2 - z_1)}{3} K(r_0, t) \begin{bmatrix} 1 & 0{\cdot}5 \\ 0{\cdot}5 & 1 \end{bmatrix} + \frac{T_{zz}}{(z_2 - z_1)} M(r_0, t) \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[M]^{\mathrm{e}} = \frac{S(z_2 - z_1)}{3} M(r_0, t) \begin{bmatrix} 1 & 0{\cdot}5 \\ 0{\cdot}5 & 1 \end{bmatrix} \tag{17}$$

where

$$K(r_0, t) = \int_0^\infty 2\pi \frac{\partial \hat{F}(\xi, t)}{\partial \xi} \frac{\partial \hat{F}(\xi, t)}{\partial \xi} (r_0 + \xi)\,\mathrm{d}\xi$$

$$M(r_0, t) = \int_0^\infty 2\pi \hat{F}(\xi, t) \hat{F}(\xi, t)(r_0 + \xi)\,\mathrm{d}\xi \tag{18}$$

In evaluating (18), the following integrals were encountered:

$$\int_0^\infty \frac{\mathrm{d}x}{(x_0 + x)^5 \left(1 + \dfrac{a}{(x_0 + x)^2}\right)^2} = \frac{\ln(a + x_0^2)}{2a^2} - \frac{\ln x_0}{a^2} - \frac{1}{2a(a + x_0^2)}$$

$$\int_0^\infty \ln^2\left(1 + \frac{a}{(x_0 + x)^2}\right)(x_0 + x)\,\mathrm{d}x = \int_0^\infty \left[\sum_{n=1}^{\infty} \frac{2}{2n-1} \left(\frac{a}{a + 2(x_0 + x)^2} \right)^{2n-1} \right]^2 (x_0 + x)\,\mathrm{d}x \tag{19}$$

Using (19), coefficients in (18) can be expressed as

$$K(r_0, t) = 4\pi \left(\frac{1}{\ln(1+\lambda)} - \frac{1}{\ln^2(1+\lambda)} + \frac{1}{(1+\lambda)\ln^2(1+\lambda)} \right)$$

$$M(r_0, t) = \frac{2\pi\mu^2 \sum_{i=1}^{n} \mathrm{e}^{\alpha_i - \alpha_0}}{\ln^2(1+\lambda)} \tag{20}$$

and

$$\alpha_i = \ln(S^{i-1}\beta_i\mu^{(n+1-i)}r_0^{2(i-1)})$$
$$\alpha_0 = \ln(s\beta_0(1+\lambda)Sr_0^2)^{n+1})$$
$$\lambda = 2{\cdot}25T_{rr}t/Sr_0^2$$
$$\mu = 2{\cdot}25T_{rr}t \tag{21}$$

where β_i, $i = 0, \ldots, n$ are constants obtained from numerical integration. The first fourteen terms of β_i are:

$$\beta_0 = 201{,}801{,}600{\cdot}0;\ \beta_1 = 256{,}398{,}799{\cdot}0;\ \beta_2 = 3{,}100{,}653{,}258{\cdot}0;\ \beta_3 = 17{,}935{,}814{,}670{\cdot}0$$
$$\beta_4 = 65{,}135{,}363{,}930{\cdot}0;\ \beta_5 = 164{,}971{,}587{,}690{\cdot}0;\ \beta_6 = 307{,}125{,}116{,}298{\cdot}0$$
$$\beta_7 = 432{,}310{,}436{,}558{\cdot}0;\ \beta_8 = 466{,}568{,}626{,}810{\cdot}0;\ \beta_9 = 387{,}470{,}938{,}140{\cdot}0$$
$$\beta_{10} = 246{,}179{,}733{,}800{\cdot}0;\ \beta_{11} = 117{,}711{,}938{,}344{\cdot}0;\ \beta_{12} = 41{,}054{,}013{,}000{\cdot}0$$
$$\beta_{13} = 9{,}868{,}658{,}800{\cdot}0;\ \beta_{14} = 1{,}463{,}061{,}600{\cdot}0$$

Having obtained the element property matrices for both finite and infinite elements, equation (2) can now be solved using the finite difference technique to discretize the time domain:

$$(\theta\Delta t[K] + [M])\{h\}^{t+\Delta t} = ([M] - (1-\theta)\Delta t[K])\{h\}^t + \{f\}\Delta t \tag{22}$$

in which θ is the time discretization parameter. A value of $\theta = 0$ indicates a forward difference scheme, $\theta = 1$ indicates a backward-difference scheme, and $\theta = 0{\cdot}5$ corresponds to a Crank–Nicolson scheme. It should be noted that since the present infinite element is time dependent, the property matrices of the element must be calculated at every time step in order to ensure the accuracy of the numerical solutions.

4. VALIDATION

To verify the accuracy and the effectiveness of the proposed infinite element two numerical examples are presented.

In the first example, a pumping well fully intersecting a laterally extensive confined aquifer is analysed using a unit thickness of the aquifer. The near field is discretized using four-node

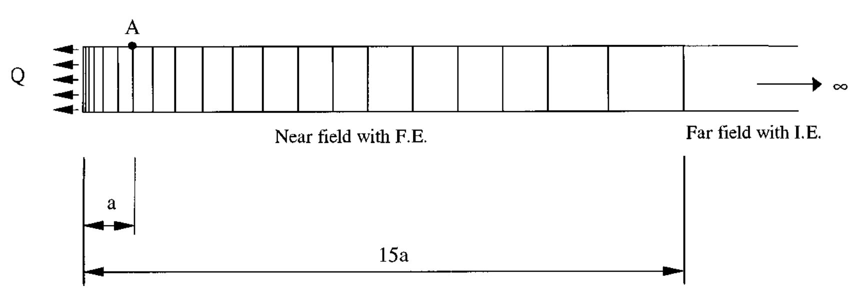

Figure 3. Pumping well fully penetrating a confined aquifer—Mesh

isoparametric finite elements and the far field is modelled using a single infinite element, as shown in Figure 3. The infinite element is placed at a distance $15a$ from the source, with a being the radial distance to the point of observation, A. Two pumping well problems are analysed: one with a constant flux, and the other with a constant head. For the sake of comparison, analyses are also

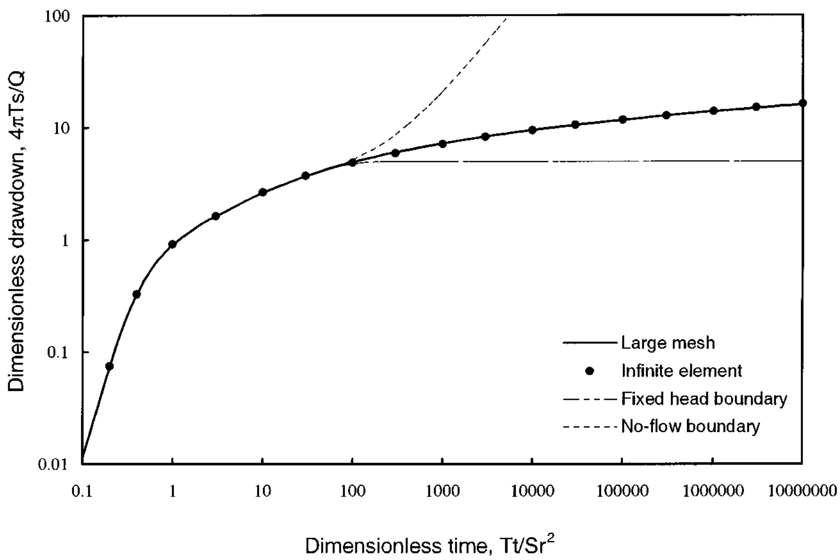

Figure 4. Time versus well drawdown at observation point A (constant-flux pumping well)

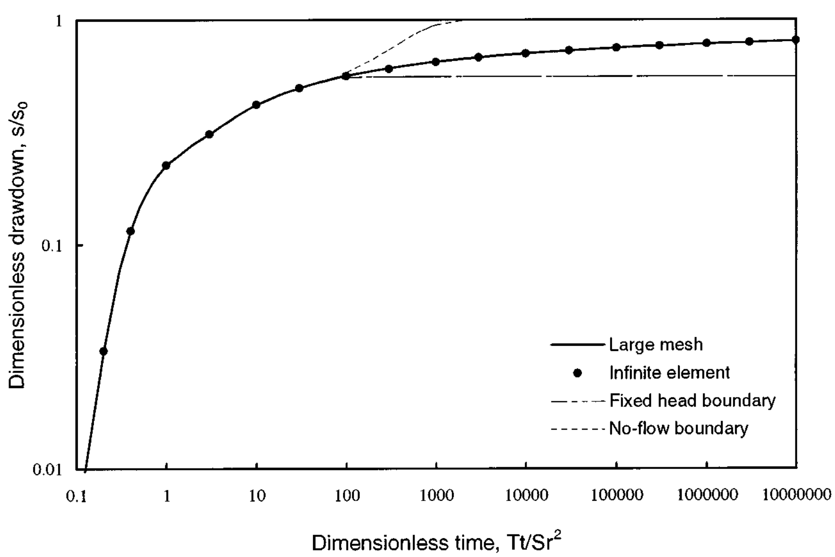

Figure 5. Time versus well drawdown at observation point A (constant-head pumping well)

performed using a large mesh extending well beyond the near field region of the aquifer (i.e. 100,000a). Analyses are also performed using conventional finite elements with fixed head or no-flow conditions imposed at the truncated near field boundary. The results of the analyses in terms of dimensionless time (Tt/Sr^2) versus dimensionless drawdown ($4\pi Ts/Q$ or s/s_0) at the observation point, A, are shown in Figures 4 and 5. As can be observed from the relevant figures, extremely good agreement exists between the results of the coupled finite-infinite elements and the results obtained using the large mesh. However, when the infinite element is removed and artificial no-flow and fixed head boundaries are introduced, the accuracy of the numerical results is affected significantly.

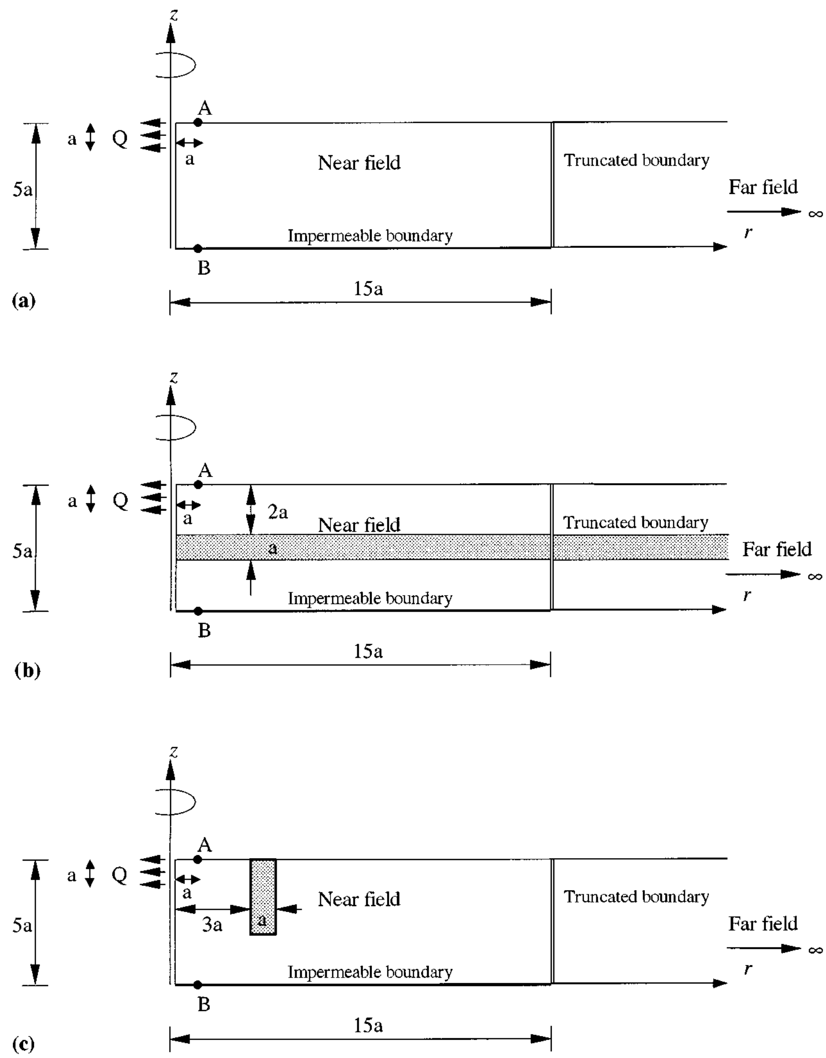

Figure 6. Pumping well partially penetrating a confined aquifer. (a) Geometry and the near field; (b) layered case; (c) with cut off wall

In the second example, the pumping well is assumed to penetrate only partially the confined aquifer. Several potential practical cases, as shown in Figure 6, are analysed: (1) a homogeneous medium, (2) a horizontally layered medium, (3) a medium with a vertical cut off wall, and (4) an anisotropic medium. The numerical results presented are obtained based on the assumption that in case (2), the permeability of the horizontal layer is 100 times the permeability of the aquifer; in

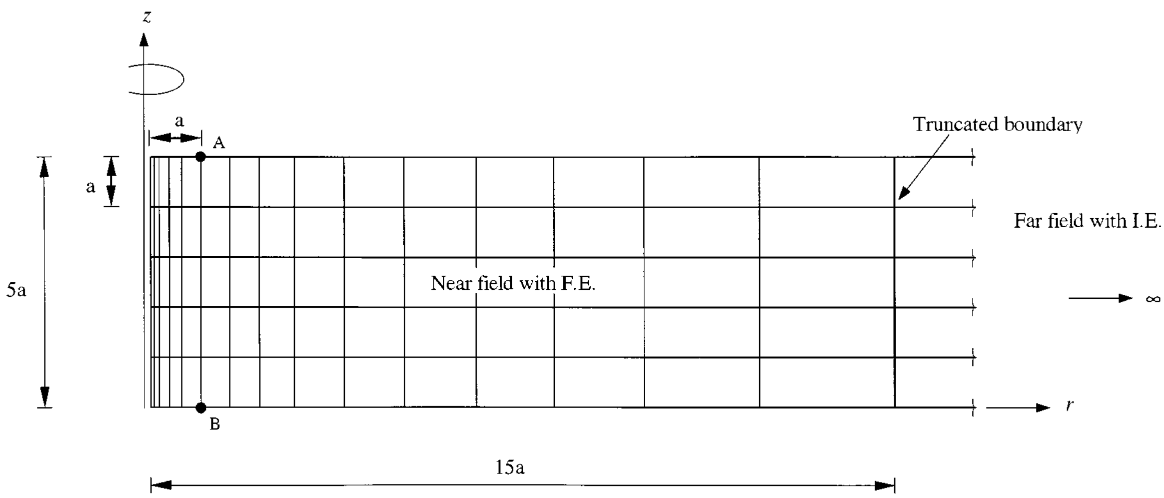

Figure 7. Pumping well partially penetrating a confined aquifer—Mesh

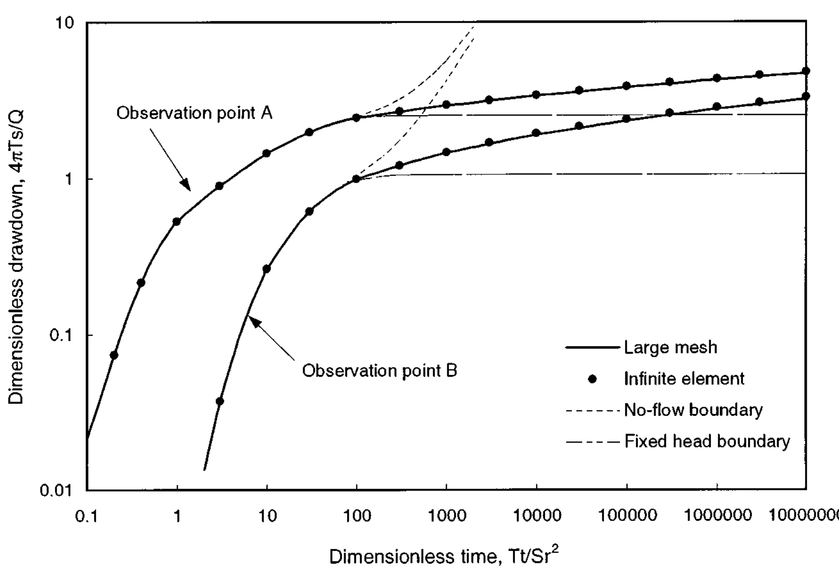

Figure 8. Time versus drawdown (homogeneous case)

case (3), the permeability of the cut off wall is two orders of magnitude smaller than that of the aquifer; and in case (4), the horizontal permeability of the aquifer is two orders of magnitude case (3), the permeability of the cut off wall is two orders of magnitude smaller than that of the aquifer; and in case (4), the horizontal permeability of the aquifer is two orders of magnitude greater than the vertical permeability of the aquifer. The mesh used in all the cases is shown in Figure 7. Four-node isoparametric finite elements are used to model the near field, and infinite elements to model the the infinite extent of the medium in the radial direction. The results of the analyses in terms of dimensionless time (Tt/Sr^2) and dimensionless well drawdown ($4\pi Ts/Q$) at observation points, A and B, are shown in Figures 8–11. Also, included in these figures are the results obtained using a large mesh of conventional finite elements extending to 100,000a. As can be observed, extremely good agreement is obtained between the results of the coupled finite-infinite elements and the large mesh. However, once the infinite elements are removed and no-flow or fixed head conditions are imposed at the truncated near field boundary, the accuracy of the numerical results deteriorates significantly, due to spurious reflections from the boundary.

5. CONCLUSIONS

An axi-symmetric infinite element has been proposed for transient analysis of flow problems in radially unbounded domains. The theoretical basis and the implementaiton of the proposed infinite element are discussed. To demonstrate the efficiency and the accuracy of the element, two pumping well problems are considered. The near field is modelled using conventional finite elements and the far field is modelled using the proposed infinite elements. It has been shown that accurate solutions can be obtained when the finite elements are coupled with the infinite elements.

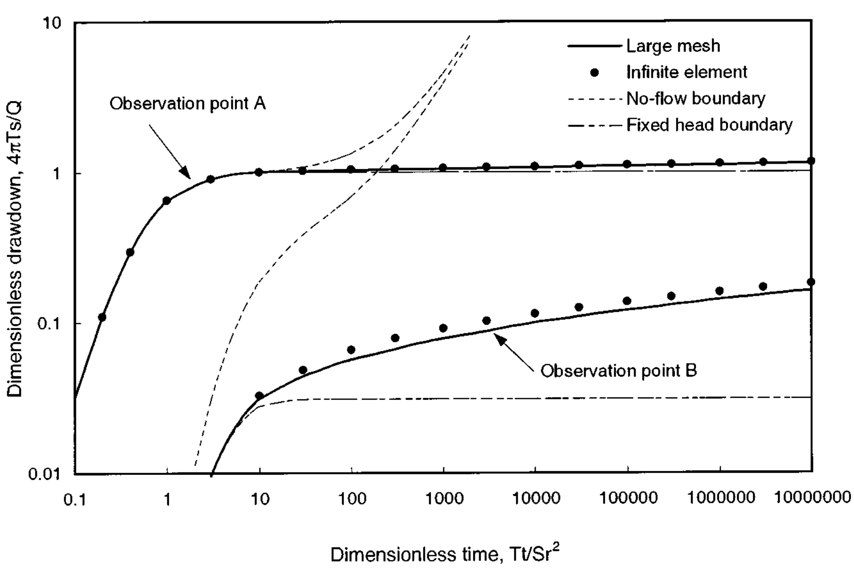

Figure 9. Time versus drawdown (layered case)

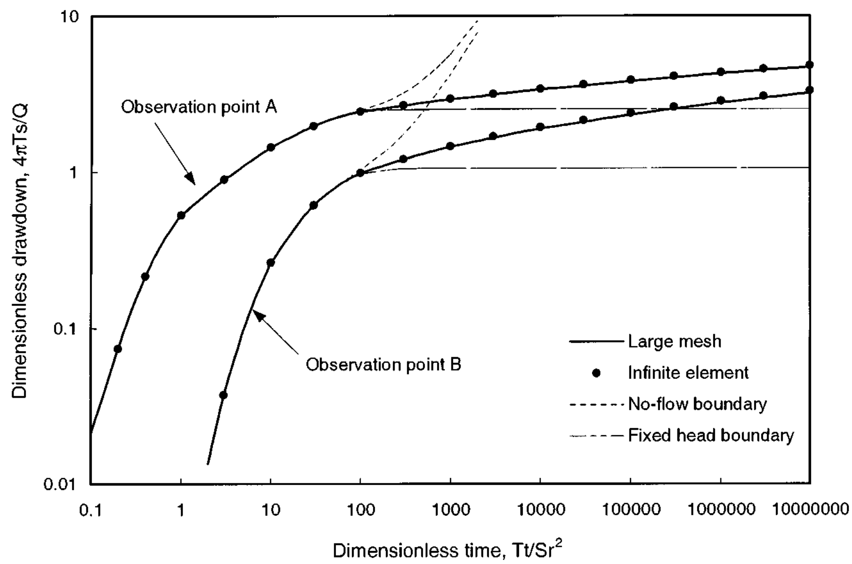

Figure 10. Time versus drawdown (with cut-off wall)

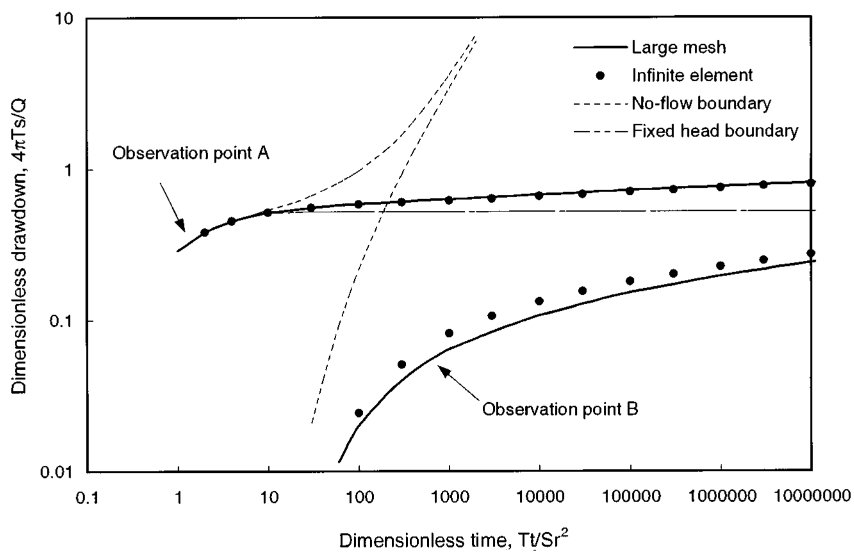

Figure 11. Time versus drawdown (anisotropic case, $T = \sqrt{Tz^z T_{rr}}$)

The accuracy of the solution significantly deteriorates when the infinite element is removed and no-flow or fixed head conditions are imposed at the truncated boundary.

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